

Clique Vectors of k -Connected Chordal Graphs

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Dedicated to Ralf Fröberg on the occasion of his 70th birthday

Abstract

The clique vector $\mathbf{c}(G)$ of a graph G is the sequence (c_1, c_2, \dots, c_d) in \mathbb{N}^d , where c_i is the number of cliques in G with i vertices and d is the largest cardinality of a clique in G . In this note, we use tools from commutative algebra to characterize all possible clique vectors of k -connected chordal graphs.

1 Introduction

The clique vector of a graph G is an interesting numerical invariant assigned to G . The study of clique vectors goes back at least to Zykov's generalization of Turán's graph theorem [8]. The clique vector of G is by definition the f -vector of its clique complex. Challenging problems including the Kalai-Eckhoff conjecture and the classification of the f -vector of flag complexes led many researchers to investigate clique vectors, see for instance [2], [3], and [6]. While the Kalai-Eckhoff conjecture is now settled by Frohmader [2], the later problem is still wide open.

Herzog et. al. [6] characterized all possible clique vectors of chordal graphs. A graph G is called k -connected if removing any set of vertices of G of cardinality less than k yields a connected graph. Thus a 1-connected

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graph is simply a connected graph. We use the convention that every graph is 0-connected. The *connectivity number* $\kappa(G)$ of G is the maximum number k such that G is k -connected. The aim of this paper is to characterize all possible clique vectors of k -connected graphs. More precisely we prove the following Theorem.

Theorem 1.1. *A vector $\mathbf{c} = (c_1, \dots, c_d) \in \mathbb{N}^d$ is the clique vector of a k -connected chordal graph if and only if the vector $\mathbf{b} = (b_1, \dots, b_d)$ defined by*

$$\sum_1^d b_i x^{i-1} = \sum_1^d c_i (x-1)^{i-1} \quad (1)$$

has positive components and $b_1 = b_2 = \dots = b_k = 1$.

The theorem above is a refinement of [6, Theorem 1.1], in the sense that putting $k = 0$, the only requirement on b -numbers is to be positive, so [6, Theorem 1.1] will be obtained.

The rest of this paper is organized as follows. In Section 2, we verify Theorem 1.1 for a subclass of chordal graphs, the so called threshold graphs, by giving a combinatorial interpretation of the b -numbers. Section 3 is devoted to a study of the connectivity of a graph via certain homological invariants of a ring associated to it. Finally, in Section 4 we prove our main theorem.

All undefined algebraic terminology can be found in the book of Herzog and Hibi [5].

2 Clique Vectors of Threshold Graphs

Let G be a graph. We denote by $S(G)$ the graph obtained from G by adding a new vertex and connecting it to all vertices of G . Also, we denote by $D(G)$ the graph obtained from G by adding an isolated vertex. Clearly the numbers of i -cliques in G and $D(G)$ are the same, unless $i = 1$. On the other hand, it is easy to verify the following formula that relates the numbers of cliques in G and $S(G)$:

$$1 + \sum_i c_i(S(G)) x^i = \left(1 + \sum_i c_i(G) x^i \right) (1 + x). \quad (2)$$

A graph T is called *threshold*, if it can be obtained from the null graph by a sequence of S - and D -operators. Thus, we have a bijection between

threshold graphs and words on the alphabet $\{S, D\}$ ending with an S^1 . Clearly, every threshold graph is chordal.

Many properties of a threshold graph can be read off from its word. Among them are the following simple but useful facts.

Lemma 2.1. *Let T be a threshold graph. Then the following hold:*

1. *The number of times that S appears in T is the clique number of T .*
2. *T is k -connected if and only if there is no D in the first k letters of T .*

□

Let T be a threshold graph with clique number d . We put a / right after every S in the word of T . Thus breaking the word of T into d subwords. Let b_i be the length of the i -th subword. Then the b -vector of T is $\mathbf{b}(T) = (b_1, b_2, \dots, b_d)$. For instance, if $T = DDDSSDSDDS$, then T breaks to $DDDS/S/DS/DDS/$ and $\mathbf{b}(T) = (4, 1, 2, 3)$.

It turns out that knowing the b -vector is equivalent to knowing the clique vector for any threshold graph.

Proposition 2.2. *Let T be a threshold graph. Then the clique vector $\mathbf{c}(T) = (c_1, \dots, c_d)$ can be obtained from $\mathbf{b}(T) = (b_1, \dots, b_d)$ using the formula*

$$\sum_{i=1}^d b_i (x+1)^{i-1} = \sum_{i=1}^d c_i x^{i-1}. \quad (3)$$

Proof. The statement is clear if T is an isolated vertex, so we may inductively assume that it has been proved for threshold graphs on $n-1$ vertices. Suppose that T is a threshold graph on n vertices. Then T is either $D(T')$ or $S(T')$, for a threshold graph T' . In the former case the statement follows easily from the induction hypothesis.

Suppose $T = S(T')$. Then $b_1 = 1$ and $\mathbf{b}(T') = (b_2, \dots, b_d)$. The induction hypothesis and equation 2 imply that

$$1 + \sum_i c_i(T) x^i = \left(1 + x \sum_{i=2}^d b_i (x+1)^{i-2} \right) (1+x).$$

Therefore

$$\sum_i c_i(T) x^i = \left(x + x \sum_{i=2}^d b_i (x+1)^{i-1} \right),$$

¹The D - and S -operations on the null graph, i.e. the graph having zero vertices, result the same graph. So, to have a unique representation of each threshold graph, we may assume that the operation D is allowed whenever the graph is not null.

as desired. \square

Let $\mathcal{B}(n, d, k)$ denote the set of all integer vectors (b_1, b_2, \dots, b_d) satisfying the following conditions:

- $b_i > 0$ for all $1 \leq i \leq d$,
- $\sum b_i = n$,
- $b_1 = b_2 = \dots = b_k = 1$.

The set of k -connected threshold graphs on n vertices and clique number d is denoted by $\mathcal{T}(n, d, k)$. The mapping $T \mapsto \mathbf{b}(T)$ is an injection from $\mathcal{T}(n, d, k)$ into $\mathcal{B}(n, d, k)$, by Lemma 2.1. A small computation, left to the reader, shows that the sets $\mathcal{T}(n, d, k)$ and $\mathcal{B}(n, d, k)$ have the same cardinality $\binom{n-k-1}{d-k-1}$. So, $T \mapsto \mathbf{b}(T)$ is indeed a bijective correspondence between $\mathcal{T}(n, d, k)$ and $\mathcal{B}(n, d, k)$. Now putting this all together, we can conclude the following characterization of clique vectors of k -connected threshold graphs.

Corollary 2.3. *A vector $\mathbf{c} = (c_1, \dots, c_d) \in \mathbb{N}^d$ is the clique vector of a k -connected threshold graph if and only if the vector $\mathbf{b} = (b_1, \dots, b_d)$ defined by equation 1 has positive components and $b_1 = b_2 = \dots = b_k = 1$.*

\square

3 Algebraic Tools

Let Γ be a simplicial complex on the vertex set $[n]$. Let \mathbb{K} be a field of characteristic zero and $R = \mathbb{K}[x_1, \dots, x_n]$ the polynomial ring on n variables. The *Stanley-Reisner ideal* I_Γ of Γ is the ideal in R generated by all monomials $x_{i_1} \cdots x_{i_l}$, where $\{i_1, \dots, i_l\}$ is not a face of Γ . The *face ring* $\mathbb{K}[\Gamma]$ of Γ is the quotient ring R/I_Γ .

Let G be a graph on the vertex set $[n]$. The collection $\Delta(G)$ of the cliques in G forms a simplicial complex, known as the *clique complex* of G . Clique complexes are *flag*, that is, all minimal non-faces have the same cardinality two. Moreover, every flag complex is the clique complex of its underlying graph (1-skeleton).

In this section we study the connectivity number of a chordal graph via a homological invariant, namely the bigraded Betti numbers (see e.g. [5, Appendix A]) of the face ring of its clique complex. We start with a general result.

Theorem 3.1. *A graph G is k -connected if and only if*

$$b_{i,i+1}(\mathbb{K}[\Delta(G)]) = 0$$

for all $i \geq n - k$. In particular,

$$\kappa(G) = \max\{k \mid b_{i,i+1}(\mathbb{K}[\Delta(G)]) = 0 \text{ for all } i \geq n - k\}. \quad (4)$$

Proof. By Hochster's formula [5, Theorem 8.1.1]

$$b_{i,i+1}(\mathbb{K}[\Delta(G)]) = \sum_{|W|=i+1} \tilde{\beta}_0(\Delta(G)_W).$$

On the other hand, the induced subcomplex $\Delta(G)_W$ is the clique complex of the induced graph G_W . So, $b_{i,i+1}(\mathbb{K}[\Delta(G)]) = 0$ if and only if G_W is connected for all W of cardinality $i+1$. Now, since the induced subgraph on a set W is the same as the graph obtained by removing the complement \overline{W} of W from G , it follows that $b_{i,i+1}(\mathbb{K}[\Delta(G)]) = 0$ if and only if removing any set of $n-i-1$ vertices results in a connected graph. Therefore $b_{i,i+1}(\mathbb{K}[\Delta(G)]) = 0$ for all $i \geq n - k$ if and only if removing any set of at most $k - 1$ vertices leaves a connected graph, as desired. \square

The Theorem above gives a general lower bound for the connectivity number of the graph.

Corollary 3.2. *If G is a graph, then $\text{depth}(\mathbb{K}[\Delta(G)]) \leq \kappa(G) + 1$.*

Proof. If the projective dimension of $\mathbb{K}[\Delta(G)]$ is p , then $b_{i,i+1}(\mathbb{K}[\Delta(G)]) = 0$ for all $i \geq p+1$. Thus, Theorem 3.1 gives the lower bound of $n-p-1$ for $\kappa(G)$. And therefore the result follows from Auslander–Buchsbaum formula [5, Corollary A.4.3]. \square

In the rest of this section, we show that the bound obtained in Corollary 3.2 is sharp as it is realized for the chordal graphs. The following fundamental result of Ralf Fröberg plays an essential role in the rest of this paper.

Theorem 3.3 (Fröberg [1]). *Let Γ be a simplicial complex. Then Γ is the clique complex of a chordal graph if and only if $\mathbb{K}[\Gamma]$ has a 2-linear resolution, i.e. $b_{i,j}(\mathbb{K}[\Gamma]) = 0$, whenever² $j - i \neq 1$.*

²Except $b_{0,0}$.

Corollary 3.4. *If G is a chordal graph, then $\text{depth}(\mathbb{K}[\Delta(G)]) = \kappa(G) + 1$.*

Proof. If G is a chordal graph, then by Fröberg’s Theorem 3.3, we have $b_{i,j}(\mathbb{K}[\Delta(G)]) = 0$, whenever $j - i \neq 1$. So, the projective dimension is equal to the maximum p such that $b_{p,p+1}(\mathbb{K}[\Delta(G)]) \neq 0$. It now follows from Theorem 3.1 that $p + 1 = n - \kappa(G)$. \square

4 Main Result

In this section, we prove our main result by using techniques from shifting theory.

A simplicial complex Γ on the vertex set $[n]$ is *shifted* if, for $F \in \Gamma$, $i \in F$, $j \notin F$ and $j < i$ the set $(F \setminus \{i\}) \cup \{j\}$ is a face of Γ . A shifted complex is flag if and only if it is clique complex of a threshold graph [7, Theorem 2]. *Exterior algebraic shifting* is an operation $\Gamma \rightarrow \Gamma^e$, associating to a simplicial complex Γ a shifted simplicial complex Γ^e , while preserving many interesting algebraic, combinatorial and topological invariants and properties. We refer the reader to the book by Herzog and Hibi [5] for the precise definition and more information. Here we mention some of the properties that will be used later on in this chapter.

Lemma 4.1. *Let Γ be a simplicial complex. Then the following hold.*

1. *Exterior shifting preserves the f -vector; $f(\Gamma) = f(\Gamma^e)$.*
2. *Alexander duality and exterior shifting commute; $(\Gamma^*)^e = (\Gamma^e)^*$.*
3. *Exterior shifting preserves the depth; $\text{depth}(\mathbb{K}[\Gamma]) = \text{depth}(\mathbb{K}[\Gamma^e])$.*

\square

The following result is known and has been used in the literature, see e.g. [4, Theorem 3.1.]. However, for the convenience of the reader, we supply a proof.

Lemma 4.2. *Let Γ be a flag complex. Then Γ is the clique complex of a chordal graph if and only if its exterior shifting Γ^e is the clique complex of a threshold graph.*

Proof. We show the “only if” direction. The other direction follows by reversing the proof sequence. Suppose that $\Gamma = \Delta(G)$ for some chordal graph G . Fröberg’s Theorem 3.3 implies that $\mathbb{K}[\Gamma]$ has a 2-linear resolution. Thus, it follows from Eagon–Reiner Theorem [5, Theorem 8.19], that the

Alexander dual Γ^* of Γ is Cohen–Macaulay of dimension $n - 3$. So, $(\Gamma^e)^*$ is Cohen–Macaulay of the same dimension, since exterior algebraic shifting commutes with Alexander duality and preserves Cohen–Macaulayness. Hence, the theorems of Eagon–Reiner and Fröberg imply that Γ^e is the clique complex of a chordal graph T . Now, since Γ^e is flag and shifted, T is a threshold graph. \square

Now we are in the position to prove our main result.

Proof of Theorem 1.1. Let G be a k -connected chordal graph. Let us denote by G^e the threshold graph such that $\Delta(G^e) = \Delta(G)^e$. It follows from part 1 of Lemma 4.1 that $\mathfrak{c}(G) = \mathfrak{c}(G^e)$. On the other hand, since $\text{depth}(\mathbb{K}[\Delta(G)]) = \text{depth}(\mathbb{K}[\Delta(G^e)])$, Corollary 3.4 implies that G^e is k -connected. Therefore the result follows from Corollary 2.3. \square

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